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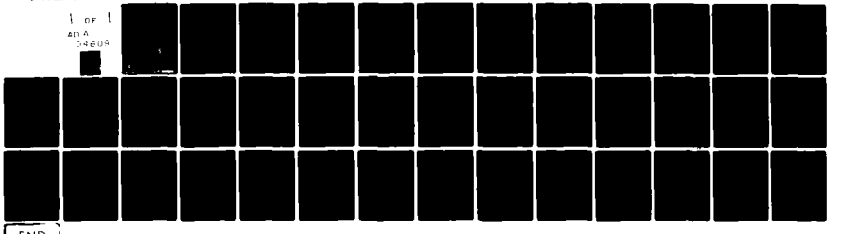

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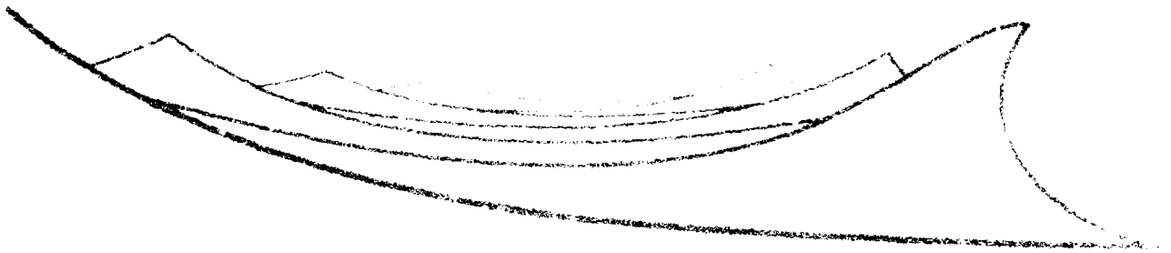

SIO REFERENCE SERIES

THE METHOD OF FRIEMAN AND KROLL FOR THE CALCULATION OF ELECTROMAGNETIC FIELDS AND ITS APPLICATION TO DATA INVERSION

Peter D. Young

University of California
September 1981

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The derivation of the method developed by Frieman and Kroll (1973) for the calculation of electromagnetic fields due to a transmitting antenna within a one-dimensional profile of electrical conductivity starts with Maxwell's equations in the charge-rationalized MKS system of units.

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Section 1

The Method of Frieman and Kroll

The derivation of the method developed by Frieman and Kroll (1973) for the calculation of electromagnetic fields due to a transmitting antenna within a one-dimensional profile of electrical conductivity starts with Maxwell's equations in the charge-rationalized MKS system of units;

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \tag{1.1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.2}$$

$$\nabla \cdot \vec{D} = \rho \tag{1.3}$$

$$\nabla \cdot \vec{B} = 0 \quad . \tag{1.4}$$

Assuming a time dependence of $e^{i\omega t}$ for the fields and currents, the first two equations become

$$\nabla \times \vec{E} = -i\omega\mu\vec{H} \tag{1.5}$$

$$\nabla \times \vec{H} = \vec{J} + i\omega\epsilon\vec{E} \quad , \tag{1.6}$$

where the linearity conditions

$$\vec{D} = \epsilon\vec{E} \tag{1.7}$$

$$\vec{B} = \mu\vec{H} \tag{1.8}$$

are assumed, with the electrical permittivity ϵ and the magnetic permeability μ taken to be independent of the field variables and not explicitly dependent on time. The choice of $e^{i\omega t}$ over $e^{-i\omega t}$ as the time dependent factor is motivated by the fact that this choice makes it possible to directly compare the field values calculated by this method with the Fourier coefficients of field value measurement real time series that have been Fourier analyzed according to the standard convention. If one then expresses the current density distribution $\vec{J}$ in the form

$$\vec{J} = \vec{J}_s + \sigma\vec{E} \quad , \tag{1.9}$$

where $\vec{J}_4$ is the current density distribution in the transmitting antenna and the other term on the right is the current density distribution induced by ohmic processes in material with an electrical conductivity distribution σ, then (1.6) becomes

$$\nabla\times\vec{H} = \vec{J}_4 + \frac{\gamma^2}{i\omega\mu}\vec{E} \tag{1.10}$$

$$\gamma^2 = i\omega\mu(i\omega\epsilon+\sigma) \quad , \tag{1.11}$$

where μ, ϵ, and σ are functions only of position in space.

A vector potential $\vec{A}$ is now defined such that

$$\vec{H} = \nabla\times\vec{A} \tag{1.12}$$

$$\vec{E} = i\omega\mu\left[\nabla\frac{1}{\gamma^2}\nabla\cdot\vec{A} - \vec{A}\right] \quad . \tag{1.13}$$

If μ is taken to be a constant, then equation (1.5) is satisfied trivially; accordingly, henceforth μ is taken to be uniformly equal to its free space value μ_0. Working with equation (1.10) gives

$$\nabla\times\vec{H} = \nabla\times(\nabla\times\vec{A}) \tag{1.14}$$

$$= \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \vec{J}_4 + \gamma^2\left[\nabla\frac{1}{\gamma^2}\nabla\cdot\vec{A} - \vec{A}\right]$$

$$\nabla^2\vec{A} - \gamma^2\vec{A} - \left[\nabla(\nabla\cdot\vec{A}) - \gamma^2\nabla\frac{1}{\gamma^2}\nabla\cdot\vec{A}\right] = -\vec{J}_4 \tag{1.15}$$

$$\gamma^2\nabla\frac{1}{\gamma^2}\nabla\cdot\vec{A} = \nabla(\nabla\cdot\vec{A}) - (\nabla\cdot\vec{A})\frac{1}{\gamma^2}\nabla(\gamma^2) \tag{1.16}$$

$$\nabla^2\vec{A} - \gamma^2\vec{A} - (\nabla\cdot\vec{A})\frac{1}{\gamma^2}\nabla(\gamma^2) = -\vec{J}_4 \quad . \tag{1.17}$$

Equations (1.12) and (1.13) admit a restricted gauge transformation for $\vec{A}$, such that $\vec{A}' = \vec{A}+\nabla\psi$ and $\vec{A}$ generate the same fields if $\nabla\psi$ satisfies equation (1.17) with $\vec{J}_4$ set uniformly equal to zero.

If ϵ and σ are taken to be functions only of the altitude coordinate z, then equation (1.17) becomes

$$\nabla^2\vec{A} - \gamma^2\vec{A} - \hat{z}(\nabla\cdot\vec{A})\frac{1}{\gamma^2}\frac{d\gamma^2}{dz} = -\vec{J}_4 \quad , \tag{1.18}$$

whose homogeneous solutions can be divided into solutions with $\vec{A}$ in the $\hat{z}$ direction and

solutions with $\vec{A}$ perpendicular to the $\hat{z}$ direction. In the first case we assume the solutions to be separated in the form

$$\vec{A} = f(z)F(x,y)\hat{z} \quad , \tag{1.19}$$

which gives from (1.18) with $\vec{J}_A$ set equal to zero

$$\frac{d^2f(z)}{dz^2} - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\frac{df(z)}{dz} - (\gamma^2+k^2)f(z) = 0 \tag{1.20}$$

$$\nabla_H^2 F(x,y) + k^2 F(x,y) = 0 \quad , \tag{1.21}$$

where the subscript H indicates that derivatives occur only in the horizontal plane. From symmetry considerations one has that the solutions arising from the separation (1.19) can give only transverse magnetic fields. Boundary conditions on f(z) may be deduced from the facts that the transverse field components $\vec{E}_T$ and $\vec{H}_T$ must be continuous across discontinuities in σ, and $\vec{E}_T$=0 at a discontinuity on one side of which σ=∞. From (1.13) we have for $\vec{E}_T$

$$\vec{E}_T = i\omega\mu\frac{1}{\gamma^2}\frac{df(z)}{dz}\nabla_H F(x,y) \quad , \tag{1.22}$$

which gives that

$$\frac{1}{\gamma_1^2}\frac{df_1(z)}{dz}\bigg|_{z_0} = \frac{1}{\gamma_2^2}\frac{df_2(z)}{dz}\bigg|_{z_0} \tag{1.23}$$

across a discontinuity in $\sigma(z)$ *at* z=z_0, and

$$\frac{df(z)}{dz}\bigg|_{z_0} = 0 \tag{1.24}$$

at a discontinuity at z=z_0 in $\sigma(z)$ on one side of which σ=∞, and from (1.12) we have

$$\vec{H}_T = f(z)\{\nabla\times F(x,y)\hat{z}\} \quad , \tag{1.25}$$

from which we have

$$f_1(z_0) = f_2(z_0) \tag{1.26}$$

across a discontinuity at z=z_0 in $\sigma(z)$. In the case of horizontal homogeneous solutions for $\vec{A}$ we assume separation in the form

$$\vec{A} = q(z)\vec{Q}(x,y) \quad , \tag{1.27}$$

where $\vec{Q}$ has no $\hat{z}$ component and we exploit the previously mentioned gauge freedom to make

the constraint

$$\nabla\cdot\vec{Q} = 0 \quad . \tag{1.28}$$

This separation gives

$$\frac{d^2q(z)}{dz^2} - (\gamma^2+k^2)q(z) = 0 \tag{1.29}$$

$$\nabla_H^2\vec{Q}(x,y) + k^2\vec{Q}(x,y) = 0 \quad . \tag{1.30}$$

Note that by (1.27) and (1.28) we have

$$\nabla\cdot\vec{A} = q(z)\nabla\cdot\vec{Q} + \vec{Q}\cdot\nabla q(z) = 0 \quad , \tag{1.31}$$

so according to (1.13) this class of solutions to (1.18) can give only transverse electric fields. For $\vec{E}_T$ we have from (1.13) and (1.27)

$$\vec{E}_T = -i\omega\mu q(z)\vec{Q}(x,y) \quad , \tag{1.32}$$

which implies that

$$q_1(z_0) = q_2(z_0) \tag{1.33}$$

across a discontinuity at $z=z_0$ in $\sigma(z)$, and

$$q(z_0) = 0 \tag{1.34}$$

at a discontinuity at $z=z_0$ in $\sigma(z)$ on one side of which $\sigma=\infty$. For $\vec{H}_T$ we have

$$\vec{H}_T = \frac{dq(z)}{dz}\{\hat{z}\times\vec{Q}(x,y)\} \quad , \tag{1.35}$$

which gives that across a discontinuity in $\sigma(z)$ at $z=z_0$

$$\left.\frac{dq_1(z)}{dz}\right|_{z_0} = \left.\frac{dq_2(z)}{dz}\right|_{z_0} \quad . \tag{1.36}$$

Mutual orthogonality of different solutions to (1.20) under identical boundary conditions is demonstrated as follows:

$$\frac{d}{dz}\left[f_m\frac{1}{\gamma^2}f_n'\right] = f_m'\frac{1}{\gamma^2}f_n' - f_m\frac{1}{\gamma^4}\frac{d\gamma^2}{dz}f_n' + f_m\frac{1}{\gamma^2}f_n'' \tag{1.37}$$

$$\frac{d}{dz}\left[f_m\frac{1}{\gamma^2}f_n' - f_n\frac{1}{\gamma^2}f_m'\right] = \frac{f_m}{\gamma^2}\left[f_n'' - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}f_n'\right] \tag{1.38}$$

$$- \frac{f_n}{\gamma^2}\left[f_m'' - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz} f_m'\right]$$

$$= \frac{f_m}{\gamma^2}(\gamma^2+k_n^2) f_n - \frac{f_n}{\gamma^2}(\gamma^2+k_m^2) f_m$$

$$= \frac{f_m f_n}{\gamma^2}(k_n^2-k_m^2)$$

$$(k_n^2-k_m^2)\int_l^u \frac{f_m f_n}{\gamma^2}\, dz = \frac{(f_m f_n'-f_n f_m')}{\gamma^2}\Big|_l^u = 0 \quad . \tag{1.39}$$

indicating that if the eigenvalues of the two solutions are different then the integral must be equal to zero. The corresponding demonstration for solutions to (1.29) is

$$\frac{d}{dz}(q_m q_n'-q_n q_m') = q_m q_n'' - q_n q_m'' \tag{1.40}$$

$$= q_m q_n (k_n^2-k_m^2)$$

$$(k_n^2-k_m^2)\int_l^u q_m q_n\, dz = (q_m q_n'-q_n q_m')\Big|_l^u = 0 \quad . \tag{1.41}$$

If m=n, it is clear that the integrals will always be positive provided that the functions within the integrals are not uniformly zero.

In anticipation of a future need, Green's functions solving the equations

$$\frac{\partial^2}{\partial z^2} g_V(z,z',k) - \frac{1}{\gamma(z)^2}\frac{d\gamma^2}{dz}\frac{\partial}{\partial z} g_V(z,z',k) \tag{1.42}$$

$$- \{\gamma(z)^2+k^2\} g_V(z,z',k) = \delta(z-z')$$

and

$$\frac{\partial^2}{\partial z^2} g_H(z,z',k) - \{\gamma(z)^2+k^2\} g_H(z,z',k) = \delta(z-z') \tag{1.43}$$

will now be constructed. A valid solution to the first of these equations is

$$g_V(z,z',k) = \frac{\bar{f}_L(z_<,k)\bar{f}_U(z_>,k)}{\gamma(z')^2 W_V(k)} \tag{1.44}$$

$$W_V(k) = \bar{f}_L(z,k)\bar{f}_U'(z,k) - \bar{f}_L'(z,k)\bar{f}_U(z,k) \quad , \tag{1.45}$$

and a valid solution to the second is

$$g_H(z,z',k) = \frac{\bar{q}_L(z_<,k)\bar{q}_U(z_>,k)}{W_H(k)} \tag{1.46}$$

$$W_H(k) = \bar{q}_L(z,k)\bar{q}_U'(z,k) - \bar{q}_L'(z,k)\bar{q}_U(z,k) \quad , \tag{1.47}$$

where $\bar{f}_L$ and $\bar{f}_U$ are solutions to (1.20) with k^2 specified and their first derivatives constrained to vanish at the lower and upper ends respectively of the range of z, $\bar{q}_L$ and $\bar{q}_U$ are solutions to (1.29) with k^2 specified and their values at the lower and upper ends respectively of the range of z constrained to vanish, $z_<$ is whichever of z and z' is the smaller, and $z_>$ is whichever of z and z' is the larger. Note that, if k_{In}^2 is one of the eigenvalues of (1.20) with all constraints enforced and and conditions imposed to require discreteness of the eigenvalues and k_{Hn}^2 is one of the eigenvalues of (1.29) under the same circumstances, g_I is singular wherever k is equal to any of the k_{In} and g_H is singular wherever k is equal to any of the k_{Hn}. These solutions to (1.42) and (1.43) are useful for practical calculation of g_I and g_H respectively, but inappropriate for the purposes of this derivation.

The derivation of the alternate form of g_I starts with the identity

$$\frac{d}{dz}\left[f_n \frac{1}{\gamma^2} g_I' - f_n' \frac{1}{\gamma^2} g_I \right] = \frac{f_n g_I}{\gamma^2}(k^2 - k_{In}^2) + \frac{f_n}{\gamma^2}\delta(z-z') \quad , \tag{1.48}$$

derived in a manner similar to that which produced (1.38), where f_n is $f_n(z)$, g_I is $g_I(z,z',k)$, γ^2 is $\gamma^2(z)$, and primes denote differentiation with respect to z. Integration of (1.48) with respect to z gives

$$0 = (k^2 - k_{In}^2) \int_l^u \frac{f_n(z) g_I(z,z',k)}{\gamma^2(z)}\, dz + \frac{f_n(z')}{\gamma^2(z')} \quad , \tag{1.49}$$

from which we have

$$g_I(z,z',k) = \frac{1}{(k_{In}^2 - k^2)} \frac{f_n(z')}{\gamma^2(z')} \frac{f_n(z)}{I_{In}} + C \tag{1.50}$$

$$I_{In} = \int_l^u \frac{f_n^2(z)}{\gamma^2(z)}\, dz \quad , \tag{1.51}$$

where C is anything that is orthogonal to $f_n(z)$ according to the definition given by (1.39); a logical guess as to the nature of C gives

$$g_I(z,z',k) = \sum_{m=0}^{\infty} \frac{1}{(k_{Im}^2 - k^2)} \frac{f_m(z')}{\gamma^2(z')} \frac{f_m(z)}{I_{Im}} \quad . \tag{1.52}$$

since from (1.39) we have

$$\int_l^u \frac{f_m(z) f_n(z)}{\gamma^2(z)}\, dz = \delta_{mn} I_{In} \quad . \tag{1.53}$$

Plugging (1.52) back into (1.42) gives

$$\sum_{n=0}^{\infty} \frac{f_n(z') f_n(z)}{\gamma^2(z') I_{In}} = \delta(z-z') \quad , \tag{1.54}$$

exactly the expression produced if one does a series expansion of $\delta(z-z')$ in the functions $f_n(z)$. The derivation of the alternate form of g_H follows the same pattern, with the result

$$g_H(z,z',k) = \sum_{m=1}^{\infty} \frac{1}{(k_{Hm}^2 - k^2)} \frac{q_m(z') q_m(z)}{I_{Hm}} \tag{1.55}$$

$$I_{Hm} = \int_l^u q_m^2(z)\, dz \quad . \tag{1.56}$$

Note that the summation in (1.52) starts with m=0, whereas the summation in (1.55) starts with m=1; due to the different boundary conditions on the solutions of (1.20) and (1.29) there is a 0*th* order solution to (1.20) whereas the lowest order solution to (1.29) is 1*st* order. Provided that the $f_n(z)$ and the $q_n(z)$ each form complete sets, which they should under the proper boundary conditions, expressions (1.52) and (1.55) should be valid solutions to (1.42) and (1.43) respectively; one condition that should give complete solution sets is that $\sigma(z)$ goes to ∞ at the upper and lower ends of the range of z.

In considering the electromagnetic fields resulting from mode excitation by a completely vertical current distribution, we define

$$\vec{J_A} = \hat{z} J_z \quad , \tag{1.57}$$

and from symmetry considerations we can write

$$\vec{A} = \hat{z} A_z \quad . \tag{1.58}$$

We define

$$A_z(\rho,\theta,z) = \sum_m \int_0^{\infty} k\, dk\, h_m(z,k) e^{jm\theta} J_m(k\rho) \quad , \tag{1.59}$$

its inverse operation

$$h_m(z,k) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} \{ \rho\, d\rho\, d\theta \tag{1.60}$$

$$\times \; A_z(\rho,\theta,z)e^{-im\theta}J_m(k\rho)\Big\} \quad ,$$

the expression

$$J_z(\rho,\theta,z) = \sum_m \int_0^\infty k\ dk\ \hat{J}_m(z,k)e^{im\theta}J_m(k\rho) \quad , \tag{1.61}$$

and its inverse

$$\hat{J}_m(z,k) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \Big\{ \rho\ d\rho\ d\theta \tag{1.62}$$

$$\times \; J_z(\rho,\theta,z)e^{-im\theta}J_m(k\rho)\Big\} \quad ,$$

where $J_m(k\rho)$ is a Bessel function of the first kind of order m. From (1.18) we have

$$\nabla^2 A_z - \gamma^2 A_z - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\frac{\partial A_z}{\partial z} = -J_z \quad ; \tag{1.63}$$

given the identity

$$\nabla^2 e^{im\theta}J_m(k\rho) = -k^2 e^{im\theta}J_m(k\rho) \quad , \tag{1.64}$$

we substitute (1.59) and (1.61) into (1.63) to get

$$\sum_m e^{im\theta}\int_0^\infty \Big\{ k\ dk\ J_m(k\rho) \tag{1.65}$$

$$\times \left[\frac{\partial^2 h_m}{\partial z^2} - (\gamma^2+k^2)h_m - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\frac{\partial h_m}{\partial z} + \hat{J}_m \right]\Big\} = 0 \quad .$$

Integration with $e^{-in\theta}$ over all θ gives

$$\int_0^\infty \Big\{ k\ dk\ J_n(k\rho) \tag{1.66}$$

$$\times \left[\frac{\partial^2 h_n}{\partial z^2} - (\gamma^2+k^2)h_n - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\frac{\partial h_n}{\partial z} + \hat{J}_n \right]\Big\} = 0$$

for every integer n; that (1.66)must hold for all positive values of ρ suggests that the expression in the brackets may be uniformly equal to zero, and indicates that even if it need not be uniformly equal to zero it can safely be constrained to be that way, as any solution for h_n that satisfies (1.66) is acceptable. Therefore, we can write

$$h_m'' - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz} h_m' - (\gamma^2+k^2) h_m = -\hat{J}_m \quad , \tag{1.67}$$

where a prime denotes partial differentiation with respect to z, and use of the Green's function g_v defined by equation (1.42) gives from this

$$h_m(z,k) = -\int_l^u g_v(z,z',k)\hat{J}_m(z',k)\, dz' \quad . \tag{1.68}$$

An alternate expression for A_z is

$$A_z(\rho,\theta,z) = \sum_m A_{zm}(\rho,z) e^{im\theta} \quad , \tag{1.69}$$

$$A_{zm}(\rho,z) = \int_0^\infty k\, dk\, h_m(z,k) J_m(k\rho) \quad ; \tag{1.70}$$

this last expression way be rewritten as

$$A_{zm}(\rho,z) = \tfrac{1}{2} \int_{-\infty}^\infty k\, dk\, h_m(z,k) H^{(2)}{}_m(k\rho) \tag{1.71}$$

by way of the identity

$$\int_{-\infty}^\infty k\, dk\, h_m(z,k) H^{(2)}{}_m(k\rho) \tag{1.72}$$

$$= 2\int_0^\infty k\, dk\, h_m(z,k) J_m(k\rho) \quad ,$$

where $H^{(2)}{}_m(x)$ is a Hankel function, composed of Bessel functions of the first and second kind according to the definitions

$$H^{(1)}{}_m(x) = J_m(x) + iN_m(x) \tag{1.73}$$

$$H^{(2)}{}_m(x) = J_m(x) - iN_m(x) \quad ; \tag{1.74}$$

the functions $H^{(2)}{}_m(k\rho)$ were chosen over the functions $J_m(k\rho)$ to describe the behavior of the A_{zm} with respect to ρ because of properties of their asymptotic behavior whose utility will become apparent below.

The identity (1.72) derives from the identities

$$J_m(-x) = (-1)^m J_m(x) \tag{1.75}$$

$$H^{(2)}{}_m(-x) = -(-1)^m H^{(2)}{}_m(x) \quad , \tag{1.76}$$

taken from Abramowitz and Stegun. The first, used with (1.60), gives

$$h_m(z,-k) = (-1)^m\ h_m(z,k) \quad , \tag{1.77}$$

from which we have by (1.76)

$$h_m(z,-k)H^{(2)}{}_m(-k\rho) = -h_m(z,k)H^{(1)}{}_m(k\rho) \quad . \tag{1.78}$$

It follows then that

$$\int_{-\infty}^{\infty} k\ dk\ h_m(z,k)H^{(2)}{}_m(k\rho) \tag{1.79}$$

$$= \int_0^{\infty} k\ dk\ h_m(z,k)H^{(1)}{}_m(k\rho) \quad ,$$

and hence

$$\int_{-\infty}^{\infty} k\ dk\ h_m(z,k)H^{(2)}{}_m(k\rho) \tag{1.80}$$

$$= \int_0^{\infty} k\ dk\ h_m(z,k)\{H^{(1)}{}_m(k\rho)+H^{(2)}{}_m(k\rho)\}$$

$$= 2\int_0^{\infty} k\ dk\ h_m(z,k)J_m(k\rho) \quad .$$

Referring back to (1.52), we see that we can rewrite (1.68) as

$$h_m(z,k) = -\sum_{n=0}^{\infty}\left\{\frac{1}{(k_{vn}^2-k^2)}\ \frac{f_n(z)}{I_{vn}}\right. \tag{1.81}$$

$$\left.\times \int_l^u \frac{f_n(z')}{\gamma^2(z')}\hat{J}_m(z',k)\ dz'\right\} \quad ,$$

which with reference to (1.62) may be expressed in the form

$$h_m(z,k) = -\frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{1}{(k_{vn}^2-k^2)}\ \frac{f_n(z)}{I_{vn}}\ K_{nm}(k) \tag{1.82}$$

$$K_{nm}(k) = \int_0^{\infty}\int_0^{2\pi}\int_l^u\left\{\rho\ d\rho\ d\theta\ dz\right. \tag{1.83}$$

$$\left.\times\ J_m(k\rho)e^{-im\theta}\frac{f_n(z)}{\gamma^2(z)}J_z(\rho,\theta,z)\right\} \quad ;$$

expression (1.71) becomes

$$A_{zm}(\rho,z) = -\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{f_n(z)}{I_{Vn}}L_{nm}(\rho) \tag{1.84}$$

$$L_{nm}(\rho) = \int_{-\infty}^{\infty} k\,dk\,\frac{K_{nm}(k)H^{(2)}{}_m(k\rho)}{(k_{Vn}^2-k^2)} \quad . \tag{1.85}$$

The Hankel function used in the integrand of this integral has the assymptotic form

$$H^{(2)}{}_m(x) \approx \sqrt{\frac{2}{\pi x}}\exp\left[-i\left\{x-\frac{m\pi}{2}-\frac{\pi}{4}\right\}\right] \quad , \tag{1.86}$$

where $x>>1$ and $x>>m$,

which suggests that (1.85) may be solved by contour integration, with the contour being drawn through the lower half plane and passing below the branch point in $H^{(2)}{}_m(k\rho)$ at k=0; the solution is

$$L_{nm}(\rho) = i\pi K_{nm}(k_{Vn})H^{(2)}{}_m(k_{Vn}\rho) \quad , \tag{1.87}$$

where k_{Vn} is the root of k_{Vn}^2 that has a negative imaginary part. Hence, we have for A_{zm}

$$A_{zm}(\rho,z) = \sum_{n=0}^{\infty} C_{nm}\,f_n(z)H^{(2)}{}_m(k_{Vn}\rho) \tag{1.88}$$

$$C_{nm} = -\frac{i}{4}\,\frac{K_{nm}(k_{Vn})}{I_{Vn}} \quad . \tag{1.89}$$

Equation (1.86) indicates that the modes which attenuate most slowly with increasing distance from the source are those that have the smallest values of $|Im(k_{Vn})|$; accordingly, the closer to the transmitting antenna the point at which one wishes to calculate the field values is, the more modes are required to give answers of acceptable accuracy. Note that (1.88) is a proper solution to the problem only if the eigenvalues are discrete.

As an example, suppose that the transmitter is a vertical point current dipole of dipole moment D, a dipole whose length dl is made arbitrarily small and the current flow I through which is made correspondingly large in such a way that the dipole moment D of the dipole, defined by

$$D = I\,dl \quad , \tag{1.90}$$

has a finite value in the limit of infinitely small length and infinitely large current; then we have for J_z

$$J_z = D\,\frac{\delta(\rho)\delta(z-z')}{2\pi\rho} \quad , \tag{1.91}$$

where z' is the antenna's vertical position and its lateral coordinate is $\rho=0$. Since J_z has no $\theta-$*dependence*, only terms of order m=0 contribute. This being the case, we immediately have from (1.83) and (1.89)

$$C_{n0} = -\frac{iD}{4}\,\frac{f_n(z')}{\gamma^2(z')I_{Vn}} \tag{1.92}$$

$$C_{nm} = 0 \quad \text{for} \quad m\neq 0 \quad . \tag{1.93}$$

In considering the case of a horizontal current distribution, we take

$$J_z = 0 \quad , \quad \vec{J}_A = \vec{J}_H \quad , \tag{1.94}$$

and assume infinitely conducting plates at the top and bottom of the range of the altitude coordinate; in order to maintain consistency with the behavior of the homogeneous solutions at such boundaries, we require that $\vec{A}_H=0$ and $A_z'=0$ at these upper and lower boundaries, constraints that serve to fix the gauge and thereby permit a unique solution for $\vec{A}$ everywhere between the plates. We define

$$\vec{A} = \vec{A}_H + \hat{z}A_z \quad , \quad \hat{z}\times\vec{A}_H=0 \quad , \tag{1.95}$$

which gives from (1.18)

$$\nabla^2\vec{A}_H - \gamma^2\vec{A}_H = -\vec{J}_H \tag{1.96}$$

$$\nabla^2 A_z - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\frac{\partial A_z}{\partial z} - \gamma^2 A_z = \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\nabla\cdot\vec{A}_H \quad . \tag{1.97}$$

We define the expressions $\vec{s}_m(z,k)$ and $\vec{J}_m(z,k)$ such that

$$\vec{A}_H(\rho,\theta,z) = \sum_m \int_0^\infty k\ dk\ \vec{s}_m(z,k)e^{im\theta}J_m(k\rho) \tag{1.98}$$

$$\vec{s}_m(z,k) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \rho\ d\rho\ d\theta\ \vec{A}_H(\rho,\theta,z)e^{-im\theta}J_m(k\rho) \tag{1.99}$$

$$\vec{J}_H(\rho,\theta,z) = \sum_m \int_0^\infty k\ dk\ \vec{J}_m(z,k)e^{im\theta}J_m(k\rho) \tag{1.100}$$

$$\vec{J}_m(z,k) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \Big\{ \rho\ d\rho\ d\theta \tag{1.101}$$

$$\times \vec{J}_H(\rho,\theta,z)e^{-im\theta}J_m(k\rho)\Big\} \quad ;$$

substitution of (1.98) and (1.100) into (1.96) with the use of the identity (1.64) gives

$$\vec{s}_m''(z,k) - (\gamma^2+k^2)\vec{s}_m(z,k) = -\vec{J}_m(z,k) \tag{1.102}$$

by the same argument that gave (1.67). Calculation of $\nabla\cdot\vec{A}_H$ from (1.98) gives

$$\nabla\cdot\vec{A}_H = \sum_m \int_0^\infty k\ dk\ \vec{s}_m(z,k)\cdot\nabla e^{im\theta}J_m(k\rho) \quad , \tag{1.103}$$

since we have

$$\nabla\cdot\vec{s}_m(z,k) = 0 \quad , \tag{1.104}$$

as $\vec{s}_m$ has no z component. This motivates the expression of A_z in the form

$$A_z(\rho,\theta,z) = \sum_m \int_0^\infty k\ dk\ \vec{p}_m(z,k)\cdot\nabla e^{im\theta}\ J_m(k\rho) \quad , \tag{1.105}$$

$$\vec{p}_m(z,k) = -\frac{1}{2\pi k^2}\int_0^{2\pi}\int_0^\infty \Big\{ \rho\ d\rho\ d\theta \tag{1.106}$$

$$\times\ A_z(\rho,\theta,z)\nabla e^{-im\theta}J_m(k\rho)\Big\} \quad .$$

Substitution of (1.103) and (1.105) into (1.97) with use of (1.64) then gives

$$\vec{p}_m''(z,k) - \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\vec{p}_m'(z,k) - (\gamma^2-k^2)\vec{p}_m(z,k) \tag{1.107}$$

$$= \frac{1}{\gamma^2}\frac{d\gamma^2}{dz}\vec{s}_m(z,k) \quad .$$

The requirements that $\vec{A}_H=0$ and $A_z'=0$ at the upper and lower boundaries places the same conditions on $\vec{s}_m$ and $\vec{p}_m$ respectively, giving $\vec{s}_m$ the same boundary conditions with respect to z as q_m and $\vec{p}_m$ as f_m.

Use of the Green's functions g_H and g_V give for $\vec{s}_m$ and $\vec{p}_m$

$$\vec{s}_m(z,k) = -\int_l^u g_H(z,z',k)\ \vec{J}_m(z',k)\ dz' \tag{1.108}$$

$$\vec{p}_m(z,k) = \int_l^u \Big\{ g_V(z,z',k) \tag{1.109}$$

$$\times \frac{1}{\gamma^2(z')} \frac{d\gamma^2(z')}{dz'} \bar{s}_m(z',k)\, dz' \quad ;$$

use of (1.55) and (1.101) in (1.108) gives for $\bar{s}_m$

$$\bar{s}_m(z,k) = -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(k_{Hn}^2 - k^2)} \frac{q_n(z)}{I_{Hn}} \bar{K}_{nm}(k) \tag{1.110}$$

$$\bar{K}_{nm}(k) = \int_l^u \int_0^{2\pi} \int_0^{\infty} \Big\{ \rho\, d\rho\, d\theta\, dz \tag{1.111}$$

$$\times\ q_n(z) e^{-im\theta} J_m(k\rho) \bar{J}_H(\rho,\theta,z) \Big\} \quad .$$

Rewriting (1.98), we have

$$\bar{A}_H(\rho,\theta,z) = \sum_{m=1}^{\infty} \bar{A}_{Hm}(\rho,z) e^{im\theta} \tag{1.112}$$

$$\bar{A}_{Hm}(\rho,z) = \int_0^{\infty} k\, dk\, \bar{s}_m(z,k) J_m(k\rho) \quad . \tag{1.113}$$

From (1.99) it is seen that $\bar{s}_m(z,k)$ has the same inversion properties with respect to k as does $J_m(k\rho)$, so by arguments similar to those used with (1.70) we have

$$\bar{A}_{Hm}(\rho,z) = \tfrac{1}{2} \int_{-\infty}^{\infty} k\, dk\, \bar{s}_m(z,k) H^{(2)}{}_m(k\rho) \quad , \tag{1.114}$$

from which we have by (1.110)

$$\bar{A}_{Hm}(\rho,z) = -\frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{q_n(z)}{I_{Hn}} \bar{L}_{nm}(\rho) \tag{1.115}$$

$$\bar{L}_{nm}(\rho) = \int_{-\infty}^{\infty} k\, dk\, \frac{\bar{K}_{nm}(k) H^{(2)}{}_m(k\rho)}{(k_{Hn}^2 - k^2)} \quad . \tag{1.116}$$

Contour integration as with (1.88) gives

$$\bar{L}_{nm}(\rho) = i\pi \bar{K}_{nm}(k_{Hn}) H^{(2)}(k_{Hn}\rho) \quad , \tag{1.117}$$

and hence

$$\bar{A}_{Hm}(\rho,z) = \sum_{n=1}^{\infty} \bar{D}_{nm}\, q_n(z) H^{(2)}{}_m(k_{Hn}\rho) \tag{1.118}$$

$$\bar{D}_{nm} = -\frac{i}{4} \frac{\bar{K}_{nm}(k_{Hn})}{I_{Hn}} \quad . \tag{1.119}$$

Work with (1.109) presents a complication, in that both g_V and $\vec{s}_m$ have numerous singularities in the complex k plane; in order for contour integration to be practical these functions must be expressed as series expansions about their singular points, but if both functions are so expressed the integration process becomes impractically cumbersome. The problem is solved by defining two versions of $\vec{p}_m$, in each of which one of the two functions is expressed in expanded form and the other is left in functional notation;

$$\vec{p}_{Hm}(z,k) = -\frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\vec{K}_{nm}(k)}{I_{Hn}} \frac{1}{(k_{Hn}^2 - k^2)} u_n(z,k) \tag{1.120}$$

$$u_n(z,k) = \int_l^u g_V(z,z',k) \frac{1}{\gamma^2(z')} \frac{d\gamma^2(z')}{dz} \times q_n(z')\, dz' \quad , \tag{1.121}$$

where (1.110), the series expansion of $\vec{s}_m$, is used, and

$$\vec{p}_{Vm}(z,k) = -\frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{\vec{M}_{nm}(k)}{I_{Vn}} \frac{1}{(k_{Vn}^2 - k^2)} f_n(z) \tag{1.122}$$

$$\vec{M}_{nm}(k) = \int_0^{\infty} \rho\, d\rho \int_0^{2\pi} d\theta \int_l^u dz \int_l^u dz' \left\{ \frac{f_n(z)}{\gamma^4(z)} \frac{d\gamma^2(z)}{dz} \times g_H(z,z',k) e^{-im\theta} J_m(k\rho) \vec{J}_H(\rho,\theta,z) \right\} \quad , \tag{1.123}$$

where (1.52), the series expansion of g_V, is used with (1.108) and (1.101). Although in principle $\vec{p}_{Vm}$ and $\vec{p}_{Hm}$ are each equivalent to $\vec{p}_m$, the standard contour integration algorithm will not see all of the singularities of either, and for purposes of contour integration they neatly divide the singularities of $\vec{p}_m$ between them; the function $\vec{p}_{Tm}$,

$$\vec{p}_{Tm} = \vec{p}_{Vm} + \vec{p}_{Hm} \quad , \tag{1.124}$$

is not equal to $\vec{p}_m(z,k)$, but within a contour integral it will behave as if it were. The expressions (1.121) and (1.123) that contain the hidden singularities are simple enough in form so that contour integration will usually not be required to evaluate them; expressions (1.44) and (1.46) may be used for the Green's functions in the integrands of these expressions.

We can now represent (1.105) in the form

$$A_z(\rho,\theta,z) = \sum_m \{A_{zVm}(\rho,\theta,z) + A_{zHm}(\rho,\theta,z)\} \tag{1.125}$$

$$A_{zVm}(\rho,\theta,z) = \int_0^\infty k\ dk\ \bar{p}_{Vm}(z,k)\cdot\nabla e^{im\theta}J_m(k\rho) \tag{1.126}$$

$$A_{zHm}(\rho,\theta,z) = \int_0^\infty k\ dk\ \bar{p}_{Hm}(z,k)\cdot\nabla e^{im\theta}J_m(k\rho)\quad , \tag{1.127}$$

with the understanding that the integrals will be evaluated with contour integration . Inspection of (1.109), (1.99), and (1.52) shows that $\bar{p}_m(z,k)$, and hence $\bar{p}_{Vm}(z,k)$ and $\bar{p}_{Hm}(z,k)$, have the same inversion properties with respect to k as $J_m(k\rho)$, so the same line of argument that gave (1.114) and (1.71) permits us to write

$$A_{zVm}(\rho,\theta,z) = \tfrac{1}{2}\int_{-\infty}^\infty \Big\{ k\ dk \tag{1.128}$$

$$\times\ \bar{p}_{Vm}(z,k)\cdot\nabla e^{im\theta}H^{(2)}{}_m(k\rho)\Big\}$$

$$A_{zHm}(\rho,\theta,z) = \tfrac{1}{2}\int_{-\infty}^\infty \Big\{ k\ dk \tag{1.129}$$

$$\times\ \bar{p}_{Hm}(z,k)\cdot\nabla e^{im\theta}H^{(2)}{}_m(k\rho)\Big\}\quad .$$

Writing these out, we have

$$A_{zHm}(\rho,\theta,z) = -\frac{1}{4\pi}\sum_{n=1}^\infty \frac{1}{I_{Hn}}\int_{-\infty}^\infty \Big\{ k\ dk \tag{1.130}$$

$$\times\ \frac{\vec{K}_{nm}(k\rho)}{(k_{Hn}^2-k^2)}u_n(z,k)\cdot\nabla e^{im\theta}H^{(2)}{}_m(k\rho)\Big\}$$

$$A_{zVm}(\rho,\theta,z) = -\frac{1}{4\pi}\sum_{n=0}^\infty \frac{1}{I_{Vn}}\int_{-\infty}^\infty \Big\{ k\ dk \tag{1.131}$$

$$\times\ \frac{\vec{M}_{nm}(k\rho)}{(k_{Vn}^2-k^2)}f_n(z)\cdot\nabla e^{im\theta}H^{(2)}{}_m(k\rho)\Big\}\quad ,$$

with only the poles explicitly shown in the denominators being evaluated. The solutions are

$$A_{zHm}(\rho,\theta,z) = \sum_{n=1}^\infty \Big\{ u_n(z,k_{Hn}) \tag{1.132}$$

$$\times\ \vec{D}_{nm}\cdot\nabla e^{im\theta}H^{(2)}{}_m(k_{Hn}\rho)\Big\}\quad .$$

where $\vec{D}_{nm}$ is as given in (1.119), and

$$A_{zVm}(\rho,\theta,z) = \sum_{n=0}^{\infty} \left\{ f_n(z) \right. \tag{1.133}$$

$$\left. \times \ \vec{B}_{nm} \cdot \nabla e^{im\theta} H^{(2)}{}_m(k_{Vn}\rho) \right\} \quad ,$$

where

$$\vec{B}_{nm} = -\frac{i}{4} \frac{\vec{M}_{nm}(k_{Vn})}{I_{Vn}} \quad . \tag{1.134}$$

A useful practical example for this work is that of a horizontal point current dipole of dipole moment M; this is represented by the expression

$$\vec{J}_H(\rho,\theta,z) = M \frac{\delta(\rho)\delta(z-z')}{2\pi\rho} \hat{h} \quad , \tag{1.135}$$

where $\hat{h}$ is a unit horizontal vector and z' is the altitude coordinate of the dipole, which is taken for convenience to have the lateral coordinate $\rho=0$. As with the vertical point dipole case, only the $m=0$ order contributes, and the appropriate coefficients are

$$\vec{D}_{nm} = -\frac{i}{4} \frac{M}{I_{Hn}} q_n(z') \hat{h} \tag{1.136}$$

$$\vec{B}_{nm} = -\frac{i}{4} \frac{M}{I_{Vn}} \int_l^u \frac{f_n(z)}{\gamma^4(z)} \frac{d\gamma^2(z)}{dz} g_H(z,z',k_{Vn})\, dz\, \hat{h} \tag{1.137}$$

$$\vec{D}_{nm} = \vec{B}_{nm} = 0 \quad \text{for} \quad m \neq 0 \quad . \tag{1.138}$$

If one's conductivity profile $\sigma(z)$ is taken to be composed of homogeneous slabs, then equations (1.20) and (1.29) have simple analytical solutions within each of the slabs, and these solutions are easily patched together at the interfaces by use of the appropriate boundary conditions. However, the solutions contain exponential factors, and as a practical consequence of this the solution process is numerically unstable. Nevertheless, excellent results were obtained by working a Ricatti transformation on the equations, and then deriving solutions to the untransformed equations from the Ricatti equation solutions. Both (1.20) and (1.29) can be represented in the form

$$y''(x) + a_1(x)y'(x) + a_0(x)y(x) = 0 \quad ; \tag{1.139}$$

the Ricatti transform is

$$v(x) = \frac{y'(x)}{y(x)} \quad , \tag{1.140}$$

and equation (1.139), transformed by this substitution, becomes the Ricatti equation

$$v'(x) + v^2(x) + a_1(x)v(x) + a_0(x) = 0 \quad . \tag{1.141}$$

From (1.140) we have that the inverse transform is

$$y(x) = C \exp\{\int^x v(t)\,dt\} \quad , \tag{1.142}$$

where C is some arbitrary constant; once a solution to equation (1.141) is obtained, the relation (1.142) can be used to give from it a solution to the equation (1.139). The most general form of the Ricatti equation is

$$v'(x) + a_2(x)v^2(x) + a_1(x)v(x) + a_0(x) = 0 \quad , \tag{1.143}$$

and it is a property of this equation that if two particular solutions to it, $v_1(x)$ and $v_2(x)$, are known, then a general solution v(x) is given by

$$\frac{v(x)-v_1(x)}{v(x)-v_2(x)} = K \exp\{\int^x a_2(t)[v_2(t)-v_1(t)]\,dt\} \quad , \tag{1.144}$$

where K is some arbitrary constant. In the event that the coefficients of (1.143) are all constants, such that

$$v'(x) + av^2(x) + bv(x) + c = 0 \quad , \tag{1.145}$$

then there are two constant solutions that can be determined by the quadratic formula;

$$v = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad . \tag{1.146}$$

Within a homogeneous slab both equations (1.20) and (1.29) give the constant coefficients

$$a=1 \quad , \quad b=0 \quad , \quad c=-(\gamma^2+k^2) \quad , \tag{1.147}$$

for which the constant solutions are

$$v_1 = -\beta \quad , \quad v_2 = \beta \quad , \quad \beta = \sqrt{\gamma^2+k^2} \quad . \tag{1.148}$$

This gives from (1.144)

$$\frac{v(x)+\beta}{v(x)-\beta} = K\, e^{+2\beta x} \quad . \tag{1.149}$$

Some algebraic manipulation of this, with appropriate choice of K, gives

$$v(x) = \frac{\beta}{\tanh\{\beta(x-x_0)+\epsilon\}} \tag{1.150}$$

$$\epsilon = \tfrac{1}{2} \log\left[\frac{v_0+\beta}{v_0-\beta}\right] \quad , \tag{1.151}$$

where

$$v_0 = v(x_0) \quad . \tag{1.152}$$

The corresponding solution for y(x), developed from (1.150) by way of (1.142), is

$$y(x) = y_0[\cosh\{\beta(x-x_0)\}+(v_0/\beta)\sinh\{\beta(x-x_0)\}] \quad , \tag{1.153}$$

where

$$y_0 = y(x_0) \quad . \tag{1.154}$$

The matching conditions at the interfaces between slabs, derived from (1.23), (1.26), (1.33), and (1.36) by way of (1.140), are

$$\frac{v_1}{\gamma_1^2} = \frac{v_2}{\gamma_2^2} \tag{1.155}$$

for the Ricatti solution to (1.20), and

$$v_1 = v_2 \tag{1.156}$$

for the Ricatti solution to (1.29). The boundary condition on v at an interface with an infinitely conducting plate on one side is v=0 for the first case, but for the second case v goes to infinity at the boundary; the approach that was used to deal with the infinitely conducting end plates was to use (1.153) in combination with either condition (1.24) or condition (1.34) to derive a value of v at the side of the end slab opposite to the side interfacing with the infinitely conducting plate. For an end slab of thickness t, the value of v at the side in question for boundary condition (1.24) is

$$v = -\beta\tanh(\beta t) \tag{1.157}$$

for the slab just below the upper end plate and

$$v = +\beta\tanh(\beta t) \tag{1.158}$$

for the slab just above the lower end plate, and for boundary condition (1.34) it is

$$\nu = -\frac{\beta}{\tanh(\beta t)} \tag{1.159}$$

for the slab just below the upper end plate and

$$\nu = +\frac{\beta}{\tanh(\beta t)} \tag{1.160}$$

for the slab just above the lower end plate. A practical limitation on this method is that it is unsafe for the end slabs to be more than about five skin depths thick, or the aforementioned numerical instability problem can render the use of (1.153) dangerously inaccurate. Given these end slab interface values, the appropriate matching conditions, and expression (1.150), a complete eigenfunction may be constructed; as only one end slab interface value is necessary for this purpose, the other can be used as a check to see if the eigenvalue has been properly determined.

An appropriately sized set of eigenvalues can be calculated to a good first approximation with a matrix approach. Equations (1.20) and (1.29) are written in the form

$$L_V\, f(z) = k_t^2 f(z) \quad , \tag{1.161}$$

$$L_V = \frac{d^2}{dz^2} - \frac{1}{\gamma^2(z)}\frac{d\gamma^2(z)}{dz}\frac{d}{dz} - \gamma^2(z)$$

$$L_H\, q(z) = k_t^2 q(z) \quad , \tag{1.162}$$

$$L_H = \frac{d^2}{dz^2} - \gamma^2(z) \quad ,$$

and the eigenfunctions f(z) and q(z) are expanded in cosine and sine series respectively so that the boundary conditions at the end plates are automatically satisfied;

$$f(z) = a_0 + \sum_{j=1}^{N} a_j \cos(j\pi z/d) \tag{1.163}$$

$$q(z) = \sum_{j=1}^{N} b_j \sin(j\pi z/d) \quad , \tag{1.164}$$

where d is the distance between the end plates and for the sake of computational practicality the series have been truncated after N terms. The coefficients are then represented as column vectors and the linear operators expressed as matrices;

$$\sum_{j=0}^{N} A_{ij} a_j = k_t^2 a_i \tag{1.165}$$

$$\sum_{l=1}^{N} B_{jl} b_l = k_H^2 b_j \tag{1.166}$$

$$A_{jl} = \frac{2}{d} \int_0^d \{L_t \cos(j\pi z/d)\} \cos(l\pi z/d)\, dz \quad , \quad l \neq 0 \tag{1.167}$$

$$A_{0l} = \frac{1}{d} \int_0^d \{L_t \cos(j\pi z/d)\}\, dz \tag{1.168}$$

$$B_{jl} = \frac{2}{d} \int_0^d \{L_H \sin(j\pi z/d)\} \sin(l\pi z/d)\, dz \quad . \tag{1.169}$$

In dealing with the L_t operator, one should note that the second term of the operator contributes an array of delta functions to the result of the operation; limit calculations indicate that the coefficient of this term is

$$\frac{1}{\gamma^2(z)} \frac{d\gamma^2(z)}{dz} = \sum_{i=1}^{N-1} \delta(z - z_i) \log\left[\frac{\gamma^2_{i+1}}{\gamma^2_i}\right] \tag{1.170}$$

where there are N homogeneous layers and z_i is the coordinate of the interface on top of layer number i. Due to the truncation of the trigonometric series representations of the eigenfunctions, the series cannot properly fit the most sharply curved of the eigenfunctions, and as a result of this there can be serious errors in the determination of their associated eigenvalues. The only effective way to deal with this problem is simply to bear in mind that some of the eigenvalues computed by the matrix method are going to be seriously in error, with those eigenvalues associated with the most quickly attenuating modes being particularly at risk, and be prepared to detect and discard the bad ones. Square matrices with dimensions on the order of 80 are easily processed for their eigenvalues by the EISPACK library of routines, developed for use with matrix eigenvalue problems and supported by many major scientific computing facilities; thanks to this library of routines it is quite practical to use outsized series expansions for the eigenfunctions in order to guarantee that a useful set of eigenvalue solutions remains after the grossly erroneous solutions have been skimmed off. The EISPACK library is described in detail by Smith et al. (1976) and Garbow et al. (1977) in the two primary users' guides for this library. It was found in practice that a useful set of eigenvalues could usually be obtained by having the number of terms in the eigenfunction expansion to be double the number of eigenvalues desired for use, thus giving double the desired number of eigenvalues, and then discarding the eigenvalues associated with the most rapidly attenuating modes until the eigenvalue set is down to the desired size. As mentioned previously, the rate of attenuation of a mode is roughly proportional to the absolute value of the imaginary part of the square root of

its eigenvalue.

As well as rendering some of the eigenvalue solutions to the matrix formulation completely mendacious, series truncation effects can easily result in minor errors in eigenvalue determination that are nevertheless sufficient to interfere with eigenfunction construction from the eigenvalues; a method is needed to detect the serious errors and refine the valid if approximate solutions. The method that was used for this purpose is to construct a solution for $v(z)$ from the top layer down to the top of the bottom layer and determine the mismatch between the value of v computed at this point from above and the value computed from below, this mismatch being negligible in magnitude if the eigenvalue used for the construction was properly determined. In the likely event that the mismatch is not of negligible magnitude, linear perturbation methods give perturbation of the mismatch value with perturbation of the proposed eigenvalue, and a version of the Newton-Ralphson method adapted for use in the complex plane may then be used to home in on the proper value of the eigenvalue in question provided that it is not too far away in the complex plane from the approximate value calculated by the matrix formulation. In the event that the trial value is not sufficiently close to the proper value to give convergence on it, the procedure will still try to find a valid solution, with the most probable outcome being a redundant convergence on some other eigenvalue. The bottom of the topmost layer could have been used instead of the top of the bottommost layer for mismatch calculation and reduction, but that interface will typically be located within a highly conducting zone modeling seawater, and it was found that a mismatch parameter calculated at a point within a highly conducting region can lead to unusably small radii of convergence in the complex k plane about the problem's eigenvalues.

In the case of a receiver located at the interface between two layers, in principle the magnetic field and the horizontal components of the electric field should be calculated as having the same values whether the receiver position is taken to be just above the interface or just below it; in practice, the method of Frieman and Kroll sometimes gives somewhat different field values for the two receiver positions, the cause of this discrepancy being traceable to a numerical problem. If, for example, the interface is between seawater and rock with an electrical conductivity many orders of magnitude lower than seawater, it turns out that the calculation of the field values at a receiver just above the interface involves taking the differences between large numbers to get a result smaller by several orders of magnitude, whereas for a receiver just beneath the interface the calculated field values are of about the same order of magnitude as the numbers subtracted from each other to get them; in the former case there is a somewhat greater opportunity for the amplification of truncation errors than is present in the latter case, and in the former case errors in the magnitudes of the calculated electric field values of at least

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50% (with a precision of 7 significant digits for a real floating point number, and with a complex number represented by two real floating point numbers) were not uncommon. Some numerical exploration of the problem was done with a two layer model with seawater on top and poorly conducting rock below, for which there are analytical expressions for the electric and magnetic fields, and agreement between these expressions and the method of Frieman and Kroll was fairly good (usually to within 5%) provided that the receiver position used with the latter was below the interface. It was noted that in cases where the conductivities of the layers above and below the interface differed by less than a couple of orders of magnitude the discrepancies between the field values calculated for the two receiver locations were usually small enough to be safely disregarded.

Section 2

Adaptation of the Method of Frieman and Kroll to Data Inversion Work

Let us suppose that for a given conductivity profile σ_0 we have, in accord with the method of Frieman and Kroll (see section 1), determined an appropriate set of eigenvalues, eigenfunctions, and Green's function component functions, and that we have used these to calculate a set of electromagnetic field values at selected points for selected frequencies. In the probable event that we find these field values calculated from the profile σ_0 to be at least mildly inconsistent with experimental measurements of the electromagnetic field, we wish to find some conductivity profile σ_1 which yields calculated field values that are consistent with the experimental measurements.

The first step in the inversion algorithm is to find some way to derive expressions of the form

$$\delta F_i = \int_l^u g_i(z)\, \delta\sigma(z)\, dz \quad , \quad i = 1 \text{ to } N \quad , \tag{2.1}$$

where F_i is one of the N calculated field values, $\delta\sigma(z)$ is a small perturbation of the conductivity profile $\sigma_0(z)$, δF_i is the accompanying change in F_i, and $g_i(z)$ is an integration kernal to be determined. The assumptions of vertical stratification and horizontal isotropy, implicit in the method of Frieman and Kroll, are retained, and imposition of the simplifying assumption that the conductivity profile is composed of discrete homogeneous slabs turns (2.1) into

$$\delta F_i = \sum_{j=1}^{M} C_{ij}\, \delta\sigma_j \quad , \quad i = 1 \text{ to } N \quad , \tag{2.2}$$

where there are M layers in the profile. Local linearity is assumed in both (2.1) and (2.2), as global linearity between σ and the F_i does not exist.

The field values F_i are calculated from a collection of eigenvalues, eigenfunctions, and Green's functions, which in turn derive from the solutions of the equations (1.20) and (1.29),

$$\frac{d^2 f(z)}{dz^2} - \frac{1}{\gamma^2(z)} \frac{d\gamma^2(z)}{dz} \frac{df(z)}{dz} - \{\gamma^2(z)+k^2\} f(z) = 0 \tag{2.3}$$

$$\frac{d^2q(z)}{dz^2} - \{\gamma^2(z)+k^2\}q(z) = 0 \tag{2.4}$$

$$\gamma^2(z) = i\omega\mu_0\sigma(z) \tag{2.5}$$

under various boundary conditions. Within a homogeneous slab both (2.3) and (2.4) have the form

$$f''(z) - (\gamma^2+k^2)f(z) = 0 \quad ; \tag{2.6}$$

if we assume f, γ^2, and k^2 to be the perturbed eigenfunction, conductivity profile function, and eigenvalue respectively, and make the substitutions

$$f(z) = f_0(z) + \delta f_0(z) \quad , \tag{2.7}$$

$$\gamma^2(z) = \gamma_0^2(z) + \delta\gamma_0^2(z) \quad ,$$

$$k^2 = k_0^2 + \delta k_0^2 \quad ,$$

where f_0, γ_0^2, and k_0^2 are the unperturbed eigenfunction, profile function, and eigenvalue respectively, we get

$$f_0'' - (\gamma_0^2+k_0^2)f_0(z) + \delta f_0''(z) - (\gamma_0^2+k_0^2)\delta f_0(z) \tag{2.8}$$

$$- (\delta\gamma_0^2+\delta k_0^2)f_0(z) - (\delta\gamma_0^2+\delta k_0^2)\delta f_0(z) = 0 \quad .$$

Assuming that the perturbation is small enough so that it is safe to discard the second order term, we then have

$$\delta f_0''(z) - (\gamma_0^2+k_0^2)\delta f_0(z) = (\delta\gamma_0^2+\delta k_0^2)f_0(z) \quad , \tag{2.9}$$

since in order for $f_0(z)$ to be a proper unperturbed solution we must have

$$f_0''(z) - (\gamma_0^2+k_0^2)f_0(z) = 0 \quad . \tag{2.10}$$

The general solution to this equation is

$$\delta f_0(z) = A\cosh(K_0z) + B\sinh(K_0z) \tag{2.11}$$

$$+ (\delta\gamma_0^2+\delta k_0^2)\, g(z) \quad ,$$

where A and B are undetermined constant coefficients,

$$K_0^2 = \gamma_0^2 + k_0^2 \quad , \tag{2.12}$$

and g(z) is a particular solution to the equation

$$g''(z) - K_0^2\, g(z) = f_0(z) \quad . \tag{2.13}$$

Solutions to (2.10) are of the form

$$f_0(z) = C\cosh(K_0 z) + D\sinh(K_0 z) \quad , \tag{2.14}$$

where C and D are undetermined constant coefficients, and substitution of (2.14) into (2.13) gives as an acceptable solution to (2.13) by standard mathematical techniques ;

$$g(z) = C\left[\frac{z}{2K_0}\sinh(K_0 z)\right] \tag{2.15}$$

$$+ D\left[\frac{z}{2K_0}\cosh(K_0 z) - \frac{1}{2K_0^2}\sinh(K_0 z)\right] \quad .$$

Some straightforward if tedious algebra gives for $\delta f_0(z)$ from (2.11)

$$\delta f_0(z) = E\,\delta f_0(z_0) + F\,\delta f_0'(z_0) + (\delta\gamma_0^2+\delta k_0^2)\,G \tag{2.16}$$

$$\delta f_0'(z) = H\,\delta f_0(z_0) + I\,\delta f_0'(z_0) + (\delta\gamma_0^2+\delta k_0^2)\,J \tag{2.17}$$

$$E = \cosh\{K_0(z-z_0)\} \tag{2.18}$$

$$F = (1/K_0)\sinh\{K_0(z-z_0)\} \tag{2.19}$$

$$G = g(z) - E\,g(z_0) - F\,g'(z_0) \tag{2.20}$$

$$H = K_0\sinh\{K_0(z-z_0)\} = K_0^2\,F \tag{2.21}$$

$$I = \cosh\{K_0(z-z_0)\} = E \tag{2.22}$$

$$J = g'(z) - H\,g(z_0) - I\,g'(z_0) \quad , \tag{2.23}$$

where z_0 is within the same slab as z. These relations, with the appropriate boundary conditions, can be used to construct $\delta f_0(z)$ and solve for δk_0^2 in terms of the conductivity profile perturbations $\delta\sigma_j$ for the M homogeneous layers considered to make up the profile. It is helpful that, although special methods must be used to avoid precision problems in the determination of f_0, no such difficulties arise in the calculation of δf_0 from f_0 once f_0 has been determined.

That both the perturbed and unperturbed eigenfunctions must satisfy the same matching conditions at the layer boundaries allows us to deduce matching conditions for the perturbation function. From the boundary conditions (1.33), (1.34), and (1.36) we have that perturbation functions on solutions to (2.4) must satisfy the conditions at a layer interface at $z=z_0$

$$\delta q_{01}(z_0) = \delta q_{02}(z_0) \tag{2.24}$$

$$\delta q_{01}'(z_0) = \delta q_{02}'(z_0) \quad , \tag{2.25}$$

and at a boundary at $z=z_0$ between a layer of finite conductivity and a layer of finite thickness and infinite conductivity

$$\delta q_0(z_0) = 0 \quad . \tag{2.26}$$

From the boundary conditions (1.26) and (1.24) respectively we have that perturbation functions on solutions to equation (2.3) must satisfy

$$\delta f_{01}(z_0) = \delta f_{02}(z_0) \tag{2.27}$$

at an interface at $z=z_0$ between two layers of finite conductivity, and

$$\delta f_0'(z_0) = 0 \tag{2.28}$$

for a boundary at $z=z_0$ with a layer of finite thickness and infinite conductivity. From boundary condition (1.23) we have for the unperturbed and perturbed functions

$$\frac{f_{01}'(z_0)}{\gamma_{01}^2} = \frac{f_{02}'(z_0)}{\gamma_{02}^2} \tag{2.29}$$

$$\frac{f_{01}'(z_0) + \delta f_{01}'(z_0)}{\gamma_{01}^2 + \delta\gamma_{01}^2} = \frac{f_{02}'(z_0) + \delta f_{02}'(z_0)}{\gamma_{02}^2 + \delta\gamma_{02}^2} \quad ; \tag{2.30}$$

using the binomial expansion and discarding terms of second order and higher gives from (2.30)

$$\frac{f_{01}'}{\gamma_{01}^2} + \frac{\delta f_{01}'}{\gamma_{01}^2} + \frac{f_{01}'\delta\gamma_{01}^2}{\gamma_{01}^4} \tag{2.31}$$

$$= \frac{f_{02}'}{\gamma_{02}^2} + \frac{\delta f_{02}'}{\gamma_{02}^2} + \frac{f_{02}'\delta\gamma_{02}^2}{\gamma_{02}^4} \quad ,$$

and use of (2.29) then gives

$$\delta f_{02}' = \frac{\gamma_{02}^2}{\gamma_{01}^2}\,\delta f_{01}' - \frac{\gamma_{02}^2}{\gamma_{01}^2}\,f_{01}'\left[\frac{\delta\gamma_{02}^2}{\gamma_{02}^2} - \frac{\delta\gamma_{01}^2}{\gamma_{01}^2}\right] \quad , \tag{2.32}$$

or, alternatively,

$$\delta f_{02}' = \frac{\gamma_{02}^2}{\gamma_{01}^2}\,\delta f_{01}' - f_{02}'\left[\frac{\delta\gamma_{02}^2}{\gamma_{02}^2} - \frac{\delta\gamma_{01}^2}{\gamma_{01}^2}\right] \quad . \tag{2.33}$$

If for a solution to (2.3) with its boundary and matching conditions δf_{00} is the perturbation function value at the interface with the lower infinitely conducting end plate (the assumption of infinitely conducting end plates capping a conductivity profile of finite thickness being carried over from section 1) and δf_{0M} is the value at the interface with the upper end plate, then by use of the above perturbation function continuation and boundary conditions we get an expression of the form

$$\delta f_{0M}' = C_0\,\delta f_{00} + \sum_{i=1}^{M} C_i\,\delta\gamma_{0i}^2 + C_{M+1}\,\delta k_0^2 = 0 \quad . \tag{2.34}$$

As a practical measure it was desirable to normalize the solutions to (2.4) such that

$$<f,f> = 1 \quad , \tag{2.35}$$

where the inner product has the definition

$$<f_1,f_2> = \int_l^u \frac{f_1(z)f_2(z)}{\gamma^2(z)}\,dz \quad , \tag{2.36}$$

and requiring (2.35) to hold for both the unperturbed and perturbed eigenfunctions gives the condition (again discarding terms of second order)

$$<f_0,\delta f_0> = 0 \quad ; \tag{2.37}$$

use of appropriate continuation and boundary conditions and some straightforward integration gives from this an expression of the form

$$<f_0,\delta f_0> = D_0\,\delta f_{00} + \sum_{i=1}^{M} D_i\,\delta\gamma_{0i}^2 + D_{M+1}\,\delta k_0^2 = 0 \quad , \tag{2.38}$$

which may be solved simultaneously with (2.34) to yield expressions for δk_0^2 and δf_{00} in terms of the layer conductivity perturbations alone; for example, δk_0^2 may be expressed in the form

$$\delta k_0^2 = \sum_{i=1}^{M} K_i\,\delta\sigma_i \quad , \tag{2.39}$$

where the K_i are complex constant coefficients. These in turn may be used with the continuation and boundary conditions to give an expression for δf_0 of the form

$$\delta f_0(z) = \sum_{i=1}^{M} L_i(z)\,\delta\sigma_i \quad . \tag{2.40}$$

The determination of the perturbation functions to solutions of equation (2.4) with its associated boundary conditions is handled in a similar manner.

As a practical matter in the simultaneous solution of equation pairs such as (2.34) and (2.38), it was found that a straightforward approach of scaling followed by elimination of a variable through subtraction frequently lead to precision problems; for example, often one coefficient would be many orders of magnitude greater than the others in each of the two equations, and it would be the equivalent coefficient in both equations. In practice, it was found that precision problems of this type occurred very seldom, if ever, in work with perturbations to solutions to equation (2.4) with associated boundary conditions, but were seldom avoided in work with equation (2.3) with associated boundary conditions. In the latter case, it was found that if one multiplied the upper end plate boundary condition equation, of which equation (2.34) is an example, by the factor F,

$$F = \frac{f_{0M}}{\gamma_0^2(\gamma_0^2+k_0^2)} \quad , \tag{2.41}$$

subtracted the result from the norm condition equation, of which (2.38) is an example, and used the result in the simultaneous solution in place of the norm equation, precision errors were always avoided. The functional form of the factor F was determined by inspection of the coefficients in the norm equation of vanishing terms in $\delta f_{0M}'$, and the numerical factor of unity was established by inspection of diagnostic printouts of the simultaneous solution routine.

In the calculation of the Green's function component functions (see for example equations (1.44) and (1.46)) it was found desirable as a practical measure to normalize each component function so that the function value was equal to unity at the layer interface nearest the end plate boundary from which the function was developed; this normalization condition together with the appropriate end plate boundary condition and the appropriate value of k^2 was sufficient to uniquely determine each component function. The requirements that a perturbed component function must also satisfy the condition at the end plate boundary and be equal to unity at the normalization interface are sufficient to constrain the perturbation function to the form

$$\delta f_c(z) = C_0\,\delta k^2 + \sum_{i=1}^{M} C_i(z)\,\delta\sigma_i \quad , \tag{2.42}$$

and, since the appropriate value of k^2 is always an eigenvalue of some eigenfunction solution of (2.3) or (2.4), we will always have some expression of the form (2.39) to substitute into (2.42) to give a solution of the form

$$\delta f_c(z) = \sum_{i=1}^{M} D_i(z)\ \delta\sigma_i \quad . \tag{2.43}$$

The propagation of the layer conductivity perturbations from this point on to the field value determinations may be straightforwardly accomplished by a notational convenience; we define a vector-like construction such that

$$A = (A_0, A_1, A_2, \cdots, A_M) = A_0 + \sum_{i=1}^{M} A_i\ \delta\sigma_i \quad , \tag{2.44}$$

with addition, subtraction, multiplication, and division defined such that

$$A + B = (A_0+B_0, A_1+B_1, A_2+B_2, \cdots, A_M+B_M) \tag{2.45}$$

$$A - B = (A_0-B_0, A_1-B_1, A_2-B_2, \cdots, A_M-B_M) \tag{2.46}$$

$$A \cdot B = (A_0B_0, A_0B_1+B_0A_1, A_0B_2+B_0A_2, \cdots, A_0B_M+B_0A_M) \tag{2.47}$$

$$\frac{A}{B} = \left(\frac{A_0}{B_0}, \frac{A_1}{B_0} - \frac{A_0B_1}{B_0^2}, \frac{A_2}{B_0} - \frac{A_0B_2}{B_0^2}, \cdots, \frac{A_M}{B_0} - \frac{A_0B_M}{B_0^2} \right) \quad . \tag{2.48}$$

The multiplication and division rules are derived from the binomial expansion, with terms of second order or higher discarded; note that it is a property of this notation that multiplication and division are exact inverse operations for each other, as they are for normal arithmetic. Replacement in the field value computation procedure of scalar quantities with the corresponding vector-like constructions will result in field value expressions of the form

$$F_i = (F_{i0}, F_{i1}, F_{i2}, \cdots, F_{iM}) \tag{2.49}$$

$$= F_{i0} + \sum_{j=1}^{M} F_{ij}\ \delta\sigma_j \quad .$$

The electric field measurements made in the experiment that is the subject of this work are most easily represented as data points on a number of complex planes; unfortunately, due to experimental difficulties, absolute phases of the data points and some of the relative phases were for practical purposes undetermined, so that field value expressions of the nature of (2.49) could not be used directly for data inversion work on the available data. Instead, an inversion procedure was constructed that worked with the lengths of vectors connecting selected pairs of data points in their complex planes,

$$r_{ij} = |F_i - F_j| \quad , \tag{2.50}$$

a choice of parameterization that adequately expresses the available experimental information

while being unaffected by the phase uncertainty problem. The notation scheme given by equations (2.44) through (2.48) in combination with the standard Pythagorean formula for distance between two points straightforwardly gave from field value expressions of the nature of (2.49) a number of vector length perturbation expressions of the form

$$l_i = l_{i0} + \sum_{j=1}^{M} D_{ij}\,\delta\sigma_j \quad , \tag{2.51}$$

which could then be used for the inversion work.

A practical difficulty that arose at this point was the fact that there were 24 vectors whose lengths were subject to fitting, while there were fewer than 10 layers in the conductivity profile model whose conductivities were subject to perturbation in order to adjust the calculated vector lengths; a least-squares approach was indicated. The problem was expressed in the form of the equation system

$$d_i = \sum_{j=1}^{M} A_{ij}\,\delta\sigma_j \quad , \quad i = 1 \text{ to } N \quad , \tag{2.52}$$

where

$$d_i = w_i(l_{ie} - l_{i0}) \tag{2.53}$$

and

$$A_{ij} = w_i D_{ij} \quad , \tag{2.54}$$

N is the number of vector lengths to be fitted, l_{ie} is the experimental value for the i *th* vector length, l_{i0} and D_{ij} are as in equation (2.51), and w_i is a weighting factor to determine the degree of emphasis of each equation in the least-squares procedure. The weighting factors were chosen to be proportional to the standard deviations of their corresponding vector lengths. the philosophy being that in the fitting procedure those vector lengths that could stand to have the greatest liberties taken with them are those that are the most poorly determined in the first place. In matrix notation we can express the equation system (2.52) as

$$d = A\,\delta\sigma \quad , \tag{2.55}$$

where d is a column vector of dimension N, $\delta\sigma$ is a column vector of dimension M, and A is an N X M matrix. The least-squares inversion of (2.55), which is designed to find the $\delta\sigma$ which minimizes the norm of the difference between the right and left sides of (2.55), employs the method of singular value decomposition, which factors the matrix A into the form

$$A = U\ Q\ V^T \quad , \tag{2.56}$$

where U is an N X M matrix such that

$$U^T\ U = I \quad , \tag{2.57}$$

V^T is an M X M matrix such that

$$V^T\ V = V\ V^T = I \quad , \tag{2.58}$$

and Q is an M X M matrix all of whose nondiagonal elements are equal to zero. We then have from (2.55) and (2.56)

$$\delta\sigma = V\ R\ U^T\ d \quad , \tag{2.59}$$

where R is the matrix obtained from Q by replacing the nonzero diagonal elements by their reciprocals. In order to avoid serious injury to the linearity approximation that underlies most of the work leading up to (2.52), the column vector d is scaled down to such a size that none of the layers of the conductivity profile model have their conductivities changed by more than 10%, a restriction that was found in practice to be sufficient to avoid divergence of the inversion procedure due to failure of the linearity approximation.

A useful feature of the method of singular value decomposition is that it can be made insensitive to machine precision error and other sources of computational noise. In a problem such as is represented by the matrix equation (2.55), there will often be one or more conductivity perturbation profiles $\delta\sigma$ that, if fed through A, will yield column vectors d of small or negligible magnitude. In such a case, when the problem is turned around and a data vector d is input to yield a perturbation profile $\delta\sigma$, noise in the data or, in extreme cases, truncation error in the machine that processes the data, can result in one or more of these null perturbation profiles being grossly exaggerated (to a degree roughly proportional to the negligibility of its associated d vector's magnitude) and superposed on the desired solution profile; this noise amplification effect can badly obscure the physical content of the problem, and it is often desirable to completely filter such null perturbation profiles from the answer. The matrix decomposition scheme given by equations (2.56) through (2.58) points out the null profiles buried in the matrix A and makes it easy to eliminate their effects on the solution profile. In this scheme, U and V are basically coordinate transform matrices, and each of the diagonal elements of the matrix Q stands for one of a number of mutually orthogonal perturbational profiles; if such an element is very small relative to the diagonal element of largest magnitude, the profile it represents will be one of the troublesome profiles. If it is judged that this profile's contribution to the solution profile obscures more than it enlightens, then its corresponding element in Q

can be set to zero and the trouble profile will be deleted from the basis set of profiles from which the solution profile is constructed. A discussion of the method and some computer routines that implement it are given in Wilkinson and Reinsch (1971). Although the need for the filtering capability of this method never arose in the course of the inversion work, it was reassuring to have this capability present in the inversion routines.

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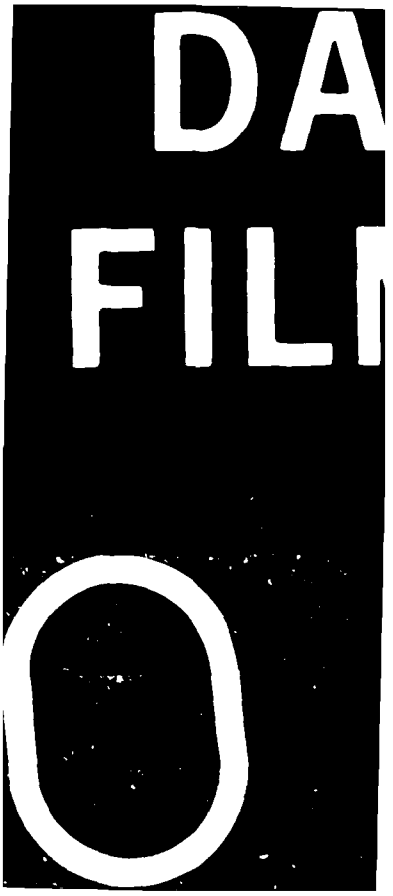
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